Dynamic Bases and Periodic Representations of Rational Numbers

Part I: Fundamentals and Problem Statement

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1. Starting Point and Motivation

The representation of rational numbers in positional numeral systems is a classical topic in mathematics. Typically, a fixed base $b \in \mathbb{N}, b \geq 2$ is assumed, and the representation of a rational number is analyzed relative to this base. In particular, periodic representations are regarded as a characteristic property of rational numbers whose denominator does not consist solely of the base's prime factors.

This perspective implies several unspoken assumptions:

- The base is a fixed, global quantity.
- The digit resolution is atomic (one digit per step).
- The period length is an intrinsic property of the number.

This work demonstrates that none of these assumptions is necessary, and abandoning them leads to a significantly more general and precise theory of periodic representations.

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2. Classical Definition of Periodicity

Let a/n be a rational number in lowest terms, and $b \geq 2$ a fixed base. The representation of a/n in base b is either finite or periodic.

In the periodic case, there exists a minimal natural number L such that

$$b^L \equiv 1 (\mathrm{mod} n')$$

where n' is the part of n coprime to b.

The number L is called the period length. This definition is correct but incomplete, as it implicitly assumes that the base and digit resolution are fixed.

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3. Group-Theoretic Interpretation

For the case of a prime denominator p with $\gcd(b, p) = 1$, a more precise description arises.

The nonzero residues modulo p form the multiplicative group

$$\mathbb{Z}_p^*.$$

The period length of the representation of $1/p$ in base b equals the order of b in this group:

$$L = \mathrm{ord}_p(b).$$

The periodic digit sequence is thus a projection of the orbit

$$1, b, b^2, b^3, \ldots (\mathrm{mod} p)$$

onto a digit alphabet.

Already here, it becomes evident that periodicity is not directly a property of the number $1/p$but a property of the group action induced by the chosen base.

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4. Block Representations and Extended Bases

Classical representations implicitly use the smallest possible resolution: one digit per step. This choice, however, is not mandatory.

Instead of base b, one can use base b^m, where $m$consecutive steps of the original representation are grouped into a block. Formally, this corresponds to considering the group action

$$x \mapsto b^m x (\mathrm{mod} p)$$

instead of

$$x \mapsto bx (\mathrm{mod} p).$$

The resulting period length is then the order of b^mmodulo p, which is always a divisor of the original order. Periodicity is therefore not atomic but hierarchically decomposable.

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5. Complete Block Periodicity

A special limiting case occurs when the base equals the denominator. For a prime denominator p,

$$1/p = 0.\bar{1}_{(p)}.$$

The representation is block-periodic with period length 1. The entire cyclic structure is encoded in a single block. This representation is neither trivial nor devoid of information; it represents the maximal compression of the periodic structure.

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6. First Consequence: Period Length is Not Absolute

From the above observations, it follows immediately:

- Period length is not invariant under a change of base.
- The same rational number possesses a whole family of equivalent periodic representations.
- Long periods arise from fine resolution, short periods from coarse resolution.

Thus, period length is not a measure of inherent complexity but a measure of the chosen scale.

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7. Transition to the Dynamic Base Perspective

If the base is treated not as a fixed quantity but as a variable parameter, a representation becomes a sequence of representations over a base path:

$$b_1, b_2, b_3, \dots$$

In this framework, a number is no longer described by a single representation but by a trajectory in the space of possible representations. This viewpoint forms the basis for a reformulation of the concept of periodic representation.

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8. Definition of Dynamic Bases

A dynamic base is not a fixed number but a sequence

$$B = (b_1, b_2, b_3, \dots)$$

with $b_k \geq 2$, where each choice of base induces a new coordinate representation of the same rational number.

A representation of a rational number in this context is no longer a single object but a family of representations parametrized by the base sequence. The classical base-10 representation corresponds to the special case of a constant base sequence $b_k = 10$.

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9. Block Size as a Free Parameter

In addition to the base, the block size is a second, independent degree of freedom. For a base $b$and block size $m \geq 1$, the representation is considered in blocks of base b^mrather than individual digits of base b. Formally, this corresponds to using the group action

$$x \mapsto b^m x (\mathrm{mod} p).$$

The classical representation is the limiting case $m = 1$.

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10. Definition of the Block Period

Let $p$be prime with gcd $(b, p) = 1$. The block period $L_{(b,m)}(p)$is defined as the smallest positive integer $k$such that

$$(b^m)^k \equiv 1 (\mathrm{mod} p).$$

The block period is thus the order of b^mmodulo p. This definition generalizes the classical period length and explicitly shows its dependence on the chosen scale.

11. Hierarchy of Periods

For fixed $p$and increasing m, a natural hierarchy of periods arises:

$$L_{(b,1)}(p) \geq L_{(b,2)}(p) \geq L_{(b,3)}(p) \geq \cdots$$

Each increase in block size leads to a non-increasing period length. The period structure is thus monotonically compressible. In the limiting case, there always exists a block size for which the block period equals 1, provided $b^m \equiv 1 (\mathrm{mod} p)$.

12. Block Periodicity as Complete Structure

A block period of length 1 does not imply the disappearance of the periodic structure but its complete contraction into a single block. All cyclic information is then contained in the block's amplitude, not in the repetition length.

Hence:

- Periodicity is not synonymous with repetition length.
- Repetition length is merely a representational choice.

13. Invariant Structure under Base Changes

Certain properties remain invariant under all base and block changes:

- The rational value.
- The group structure of $\mathbb{Z}_p^*$.
- The existence of cyclic orbits.

Non-invariant aspects include:

- Digit sequences,
- Period lengths,
- Block sizes.

The fundamental structure thus lies not in the representation but in the underlying group action.

14. Order as a Scale-Dependent Quantity

The order of an element in $\mathbb{Z}_p^*$is an algebraic quantity. The observed period length, however, is its projection onto a chosen scale.

Order is therefore a scale-dependent phenomenon:

- Finer scales produce longer orbits.
- Coarser scales produce shorter orbits.

Order is thus relative to observational resolution, not absolute.

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15. Representations as Trajectories

A rational number is not described by a single representation but by a trajectory:

$$(a/p) \mapsto \{(b, m, L_{(b,m)}(p))\}.$$

This trajectory describes how the same number appears under variations of base and block size. The classical representation is merely a point on this trajectory.

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16. Consequences for the Concept of Uniqueness

Uniqueness of representation holds only under the implicit assumption of a fixed base and block size. Once this assumption is abandoned, uniqueness must be replaced by equivalence classes of representations. Two representations are equivalent if they can be transformed into one another via a base or block change.

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17. Transition to the Structural View

The preceding results show that periodic representations should not be treated as primary objects. Instead, they are secondary projections of an invariant structure. This insight underlies a structural reinterpretation of periodic representations, where complexity, order, and repetition appear as scale-dependent quantities.

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18. Need for a Redefinition of the Period

The classical definition identifies the period with the minimal repetition length of a digit sequence in a fixed base. This definition is formally correct but conceptually insufficient, as it implicitly fixes the base and resolution.

From the preceding sections, it follows that there is no distinguished, canonical period length. Instead, a family of period lengths exists, parametrized by base and block size. This necessitates a reformulation of the concept of a period.

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19. Structural Period versus Representational Period

It is proposed to distinguish two levels:

- **Structural period**
  The structural period is the order of an element in the underlying algebraic structure, e.g., the order of a generator in $\mathbb{Z}_p^*$.
- **Representational period**
  The representational period is the observed repetition length of a digit or block sequence in a chosen base and block size.

The structural period is invariant; the representational period is scale-dependent.

20. Block Periods as Quotients of the Structural Period

For a given structural period L, all possible representational periods are divisors of L, induced by appropriate choices of block size and base.

The classical period length is thus the maximal representative of this family. Shorter periods are not degeneracies but legitimate projections of the same structure.

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21. Revision of Classical Classifications

Classical concepts such as:

- **full reptend prime**,
- **cyclic number**,
- **primitive period**,

are based on the choice of a fixed base (usually base 10). In the dynamic base perspective, these concepts are not fundamental but base-dependent distinctions. A prime can have a maximal period in one base and a minimal block period in another.

Thus, these classifications lose their absolute character and become relative statements about a specific coordinatization.

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22. Families, Shifts, and Coset Structure

The decomposition of periodic representations into families and shift positions corresponds to the coset structure of the underlying group.

Each family represents a coset of a cyclic subgroup; the shift corresponds to the position within the orbit. This structure is independent of the chosen base but is revealed differently depending on base and block choice.

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23. Order as a Multi-Scale Phenomenon

Order does not appear as a singular quantity but as a multi-scale phenomenon:

- On a fine scale, it manifests as long periods.
- On a coarse scale, it collapses into short or single-block periods.

This scale-dependence explains why seemingly chaotic digit sequences can appear highly ordered under a suitable choice of base.

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24. Representation Complexity and Information Content

The length of a periodic representation is not a measure of its information content. Rather, it measures the redundancy of the representation relative to the chosen scale.

Compression via base or block changes removes representational redundancy, not structural information. Compression is therefore not merely an algorithmic process but a structural simplification.

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25. Periodic Representation as a Measurement Process

In the dynamic base perspective, a periodic representation is the result of a measurement process:

- The number is the object.
- The base is the measuring instrument.
- The digit sequence is the measurement outcome.

Different bases measure different aspects of the same structure. No single representation is privileged.

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26. Transition from Objects to Equivalence Classes

Instead of comparing individual representations, it is more meaningful to consider equivalence classes of representations that can be transformed into one another via base or block changes.

These equivalence classes represent the true mathematical structure, while individual representations are merely representatives.

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27. Consequences for Mathematical Order

The analysis suggests that order in mathematics is often not missing but obscured by inappropriate coordinatizations.

Long periods, apparent randomness, and high complexity are often artifacts of too fine or poorly chosen scales.

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28. Summary of the Structural Shift

The essential shift is to treat periodic representations not as primitive objects but as scaled projections of an invariant structure.

This shift requires no new mathematics but a more precise interpretation of known algebraic concepts.

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29. Order as a Relational Quantity

The previous results show that order cannot be understood as an absolute property of a mathematical object. Order always manifests relative to a chosen action, scale, or coordinatization.

In the context of periodic representations of rational numbers, this means:

- The underlying algebraic structure has order.
- The observed order is a projection of this structure.
- The projection depends on the base and block size.

Order is thus not a property of an isolated object but a relational quantity between object and representation mechanism.

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30. Separation of Structure and Appearance

The dynamic base perspective enforces a strict separation between:

- **Structure:** invariant under base and block changes
- **Appearance:** dependent on the chosen representation

The classical decimal period belongs entirely to the level of appearance. The cyclic group structure belongs to the level of structure.

This separation is conceptually analogous to familiar distinctions in other areas of mathematics, e.g., between coordinates and geometric objects.

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31. Representation-Dependent Mathematical Complexity

Complexity is often judged by the length or irregularity of a representation. The present analysis shows that this assessment is not invariant.

A long periodic representation can be compressed to a short block period or even a single-block representation through an appropriate choice of base, without losing structural information.

Thus, representational complexity is not intrinsic but depends on the chosen scale.

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32. Revising the Concept of "Randomness" in Digit Sequences

Periodic digit sequences of rational numbers are often described as pseudorandom, especially when the period length is maximal in base 10.

In the dynamic base perspective, this perception is an artifact:

- The same structure can appear highly regular in another base.
- Randomness arises from an inappropriate coordinatization, not from a lack of order.

Randomness is therefore not primarily a property of the number but a property of the observation scale.

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33. Dynamic Bases as an Epistemic Tool

The choice of base functions as a tool for insight. Different bases reveal different aspects of the same structure.

A dynamic base allows these aspects to be systematically revealed by controlling the scale of the representation. Representations are not replaced but contextualized.

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34. Representation as a Path in Coordinate Space

If the base is treated as a dynamic parameter, a representation is no longer a single object but a path in the space of possible coordinates.

This path describes how the same number manifests under varying resolution. The classical representation is merely a point on this path.

This viewpoint shifts focus from individual representations to transitions between representations.

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35. Consequences for the Concept of Uniqueness

Uniqueness only holds within a fixed coordinatization. When the coordinatization itself becomes variable, uniqueness must be replaced by invariance.

Two representations are not equal when they have identical digit sequences but when they carry the same structural information and can be transformed into one another via permissible transformations.

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36. More General Scope

Although the analysis was conducted using rational numbers as an example, the underlying idea is of general nature.

Whenever mathematical objects are accessed via representations, the question arises as to which properties are invariant and which are mere artifacts of the representation.

The dynamic base perspective is therefore not restricted to number representations but is a special case of a general principle.

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37. Order as an Emergent Property

From this perspective, order does not appear as a static property but as an emergent phenomenon, revealed only through the interplay of structure and representation.

Depending on the scale, the same structure can appear ordered, complex, or seemingly chaotic without the underlying structure itself changing.

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38. Consolidation of Results

The investigation shows:

- Periodic representations are scale-dependent.
- Order is relational, not absolute.
- Compression is a form of structural clarification, not information reduction.
- The classical decimal representation is a special case of maximal resolution, not the canonical state.

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39. Transition to a Re-Evaluation of Classical Concepts

The insights gained suggest a re-evaluation of central concepts in number theory and representation theory, particularly where they implicitly assume a fixed base.

This affects both pedagogical representations and theoretical classifications.